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Discussion

Discussion on "Flattening in shear zones under constant volume: a theoretical evaluation" by N. Mandal, C. Chakraborty and S. Samanta [Journal of Structural Geology 23 (2001) 1771–1780][☆]

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1. Introduction

Mandal et al. (2001) (hereafter referred to as Mandal et al.) present a theoretical analysis of flattening in shear zones under constant volume and derive relationships between S_r, the ratio of pure to simple shear rates of Ghosh and Ramberg (1976), W_k the kinematical vorticity number of Ghosh (1987), D_f, the length to width ratio of a shear zone and α, the inclination of the shear zone normal and the bulk compressive direction. Furthermore, they consider separately the cases of a shear zone with perfectly rigid and deformable boundaries. Several errors in the analysis of Mandal et al. are presented here which indicate that many of the conclusions of Mandal et al. regarding the relationship between parameters in flattening shear zones are incorrect. I use the same notation as Mandal et al. where possible, however, their velocities (u,v) are denoted by (v_x,v_y) because componential equations are used here.

2. Mandal et al.'s solution

2.1. Bulk stress field

Mandal et al. state that they are considering a shear zone whose normal makes an angle α with the principal direction of compressive stress and indicate that the normal (σ_n) and shear stress (τ) acting along the shear zone boundaries are given by (their Eqs. (A7a) and (A7b)):

$$\sigma_n = p\cos 2\alpha \tag{1}$$

$$\tau = -p\sin 2\alpha \tag{2}$$

where:

$$p = \sigma_1 - \sigma_2/2 \tag{3}$$

as defined in the appendix of Mandal et al. and σ_1 and σ_2 are the principal bulk stresses. The term p is variously described as the 'bulk differential compression' or the 'differential compressive stress'. However, differential stress is defined as the difference in magnitudes between the maximum and minimum principal stresses (Price and Cosgrove, 1990, p. 7) and is given as $\sigma_1 - \sigma_2$ in the case of plane stress. I suspect that Mandal et al. actually meant p to be the principal compressive deviatoric stress component, which is correctly given as $\sigma_1 - \sigma_2$ and in which case the expressions for normal and shear deviatoric stresses acting on the boundary are correct.

2.2. Energy calculations

One of the basic premises of Mandal et al. is that the theory is developed by "balancing the energy involved in the flow within the shear zone with the work to be done for the movement of the boundary walls". This is a statement of the first law of thermodynamics, however, heat effects are ignored and body forces (e.g. due to gravity) are assumed to be negligable. The mechanical power input (i.e. work rate) is given by (Malvern, 1969, p. 227):

$$\int_S \mathbf{F}\cdot\mathbf{v}\,dS \tag{4}$$

where $\mathbf{F}$ is the traction or force acting along a surface (S) enclosing the flow and $\mathbf{v}$ is the velocity. In three dimensions, the surface S totally encloses the volume of fluid under consideration, whereas in two dimensions S is a curve totally enclosing an area of fluid. Mandal et al. erroneously calculate the work rate for the flattening shear zone by only

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considering the work done along the upper and lower surfaces of the shear zone. They mistakenly ignore the contribution due to work done at the shear zone ends. S must be a closed surface or curve.

From the definition of the stress tensor $\mathbf{F} = \boldsymbol{\sigma}\mathbf{n}$ on S, where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{n}$ is the outward unit normal vector so that:

$$\int_S \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{v}\,\mathrm{d}S = \int_S \boldsymbol{\sigma}\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S \tag{5}$$

Applying the divergence theorem (see Schey (1997, p. 47) for example) we have:

$$\int_S \boldsymbol{\sigma}\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = \int_V \nabla\cdot(\boldsymbol{\sigma}\mathbf{v})\,\mathrm{d}V \tag{6}$$

This is the mathematical expression for the energy balance used by Mandal et al. If Eq. (6) is to be of use, we need to understand it at the component level. The lhs of Eq. (6) is:

$$\int_S \sigma_{ij}v_i n_j\,\mathrm{d}S \tag{7}$$

and the rhs is:

$$\int_V \frac{\partial\left(\sigma_{ij}v_i\right)}{\partial x_j}\mathrm{d}V \tag{8}$$

$$= \int_V \sigma_{ij}\frac{\partial v_i}{\partial x_j}\mathrm{d}V + \int_V v_i\frac{\partial \sigma_{ij}}{\partial x_j}\mathrm{d}V \tag{9}$$

where the product rule is applied to get Eq. (9) and the Einstein summation convention is used.

A few terms need to be defined. The velocity gradient tensor is:

$$\mathbf{L} = L_{ij} = \frac{\partial v_i}{\partial x_j}$$

and the stretching (or rate of deformation) tensor is:

$$\mathbf{D} = D_{ij} = \frac{1}{2}\left(L_{ij} + L_{ji}\right)$$

(Malvern, 1969, pp. 146–147). The consitutive equation of an incompressible Newtonian fluid is:

$$\sigma_{ij} = 2\eta D_{ij} - \delta_{ij}p^{*} \tag{10}$$

where η is viscosity, δ_{ij} is the Kronecker delta and p^{*} is the pressure term (usually denoted simply by p, but modified here to avoid confusion with the principal deviatoric compressive stress term of Mandal et al.) given as:

$$p^{*} = \frac{\sigma_{ii}}{m} \tag{11}$$

where m is the dimension under consideration.

Hence the first term of Eq. (9) is:

$$\sigma_{ij}\frac{\partial v_i}{\partial x_j} = \left(2\eta D_{ij} - \delta_{ij}p^{*}\right)L_{ij} \tag{12}$$

$$= 2\eta D_{ij}L_{ij} \tag{13}$$

due to incompressibility (i.e. $L_{ii} = 0$ and $D_{ii} = 0$). For two dimensions this gives:

$$\int_V \sigma_{ij}\frac{\partial v_i}{\partial x_j}\mathrm{d}V = 2\eta\int_V\left(D_{xx}^2 + D_{yy}^2 + 2D_{xy}^2\right)\mathrm{d}V \tag{14}$$

$$= 4\eta\int_V\left(D_{xx}^2 + 2D_{xy}^2\right)\mathrm{d}V \tag{15}$$

which is equivalent to the expression used by Mandal et al. to calculate the work rate inside the shear zone (i.e. their Eq. (A1)) except that they denote D_{11} by ϵ_{xx} etc. If gravitational and inertial forces are ignored then the second term of Eq. (9) evaluates to zero; however, this may not always be the case for other materials and models. The above discussion highlights that Eqs. (7) and (15) are equal and it would be surprising if application of either gave different expressions for the work rate, as appears to be the case in the analysis of Mandal et al.

In producing a model for a constant volume flattening shear zone, Mandal et al. begin with the model described by Jaeger (1969, pp. 140–143), which is also considered by Ramsay and Lisle (2000, pp. 998–999) and is referred to as the 'cream cake' model. There are a number of problems with Mandal et al.'s understanding of this model. Firstly one of Mandal et al.'s basic premises is the assumption that "the flow of material in response to flattening takes place along the shear direction", which would imply that $v_y = 0$, even though the cream-cake model clearly includes non-zero velocity components in both the x- and y-directions (see Eqs. (1a) and (1b) of Mandal et al.). In addition to the error of not considering the work rate at the shear zone ends (see above), they are also in error when they calculate the work done per unit time in bringing the rigid upper and lower walls together due to flattening.

In the model of Mandal et al., a shear zone with rigid boundaries is contained in an unspecified rigid material, which is subject to bulk principal stresses of σ_1, σ_2 and p and is oriented at α to the shear zone normal. It appears from the expression given for the work done per unit time in bringing the rigid walls together (their Eq. (A8)) that Mandal et al. are assuming that the stress responsible for the flattening deformation is constant along the shear zone boundary (i.e. the normal component of p, given by $\sigma_{\mathrm{n}} = p\cos 2\alpha$). Using Eq. (7) for the upper wall we have:

$$\int_S \sigma_{ij}v_i n_j\,\mathrm{d}S = \int_{-l}^{l} \sigma_{ij}v_i n_j\,\mathrm{d}x \tag{16}$$

$$= 2lv_b p\cos 2\alpha \tag{17}$$

where v_b is the velocity in the y-direction at $y = t$, so that the work rate for the upper and lower walls is:

$$= 4lv_b p\cos 2\alpha \tag{18}$$

and not the expression given by Mandal et al., which is twice as large. However, this error may be related to the error in defining p.

Furthermore, the assumption of Mandal et al. that the stress along the boundaries of the shear zone is constant is in error, because a constant stress distribution cannot produce the flattening flow they consider. Jaeger (1969, p. 142) gives an expression for normal stress in the cream-cake model (assuming that the pressure term (p^*) is zero at $x = \pm l$ and $y = \pm t$), it varies in both the x- and y-directions and is given by:

$$\sigma_{yy} = \sigma_n = \frac{3\eta_s v_b}{2t^3}\left[3\left(t^2 - y^2\right) + x^2 - l^2\right] \quad (19)$$

and is clearly variable in the x-direction at both the upper and lower boundaries (i.e. $y = \pm t$). It is physically impossible to have a situation where the normal stress is constant on one side of a boundary and variable on the other side as this is in violation of Newton's Third Law of Motion.

2.3. Summary

In this section I have pointed out several fundamental errors associated with the analysis of Mandal et al. They are:

1. A confusion between differential and deviatoric stress.
2. Failing to calculate the work rate along the full curve enclosing the fluid area in the shear zone.
3. Violation of Newton's Third Law by equating a constant normal stress with the variable normal stress required for the flattening flow.

3. Correct solution

In light of the errors in Mandal et al.'s solution, it is appropriate to give the correct theory here and assess whether or not relationships of the type derived by Mandal et al. can be deduced.

Jaeger (1969, pp. 140–142) gives the full solution for the flattening flow:

$$v_x = \frac{3v_b x(t^2 - y^2)}{2t^3} \quad (20)$$

$$v_y = \frac{v_b y(y^2 - 3t^2)}{2t^3} \quad (21)$$

$$\sigma_{xx} = \frac{3\eta_s v_b[3(t^2 - y^2) + x^2 - l^2]}{2t^3} \quad (22)$$

$$\sigma_{yy} = \frac{3\eta_s v_b[y^2 - t^2 + x^2 - l^2]}{2t^3} \quad (23)$$

$$\sigma_{xy} = \sigma_{yx} = \frac{3\eta_s v_b xy}{t^3} \quad (24)$$

First the work rate due to forces along the closed curve $ABCD$ (where $A = (-l,t)$, $B = (l,t)$, $C = (l,-t)$ and $D = (-l,-t)$) are calculated, letting $Q = \sigma_{ij}v_i n_j$ for conciseness:

$$\int_S \sigma_{ij}v_i n_j \mathrm{d}S = \int_A^B Q\mathrm{d}x + \int_B^C Q\mathrm{d}y + \int_C^D Q\mathrm{d}x + \int_D^A Q\mathrm{d}y \quad (25)$$

$$= \int_{-l}^{l} Q\mathrm{d}x + \int_{h}^{-h} Q\mathrm{d}y + \int_{l}^{-l} Q\mathrm{d}x + \int_{-h}^{h} Q\mathrm{d}y \quad (26)$$

by noting that:

$$\sigma_{ij}v_i n_j = \sigma_{xx}v_x n_x + \sigma_{xy}v_x n_y + \sigma_{yx}v_y n_x + \sigma_{yy}v_y n_y \quad (27)$$

(and taking into account that the velocity components are correctly signed in Jaeger's (1969) solution), we have:

$$\sigma_{ij}v_i n_j\big|_{AB} = \sigma_{yy}v_y\big|_{AB} = -\frac{3\eta_s v_b^2[x^2 - l^2]}{2t^3} \quad (28)$$

$$\sigma_{ij}v_i n_j\big|_{BC} = \left(\sigma_{xx}v_x + \sigma_{yx}v_y\right)\big|_{BC} = -\frac{3l\eta_s v_b^2[9t^4 - 12t^2y^2 + 7y^4]}{4t^6} \quad (29)$$

$$\sigma_{ij}v_i n_j\big|_{CD} = \sigma_{yy}v_y\big|_{CD} = -\frac{3\eta_s v_b^2[x^2 - l^2]}{2t^3} \quad (30)$$

$$\sigma_{ij}v_i n_j\big|_{DA} = \left(\sigma_{xx}v_x + \sigma_{yx}v_y\right)\big|_{DA} = -\frac{3l\eta_s v_b^2[9t^4 - 12t^2y^2 + 7y^4]}{4t^6} \quad (31)$$

Upon evaluating the integrals the work rate due to forces along the boundaries is:

$$\frac{4l\eta_s v_b^2[24t^2 + 5l^2]}{5t^3} \quad (32)$$

which is equivalent to the expression for work rate calculated by Mandal et al. (using their Eq. (A2) or Eq. (15) above) for inside the shear zone. Therefore the energy balance invoked by Mandal et al. is simply an equality that cannot be used to derive further information about the relationship between variables.

If this analysis is applied to the simple shear flow or to the flattening flow in a shear zone with deformable walls of Mandal et al., the result will be an equality as derived above for the flattening flow. In fact, this equality is guaranteed by the divergence theorem. Therefore, the relationships between S_r, W_k, D_f, and α derived by Mandal et al. are incorrect.

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